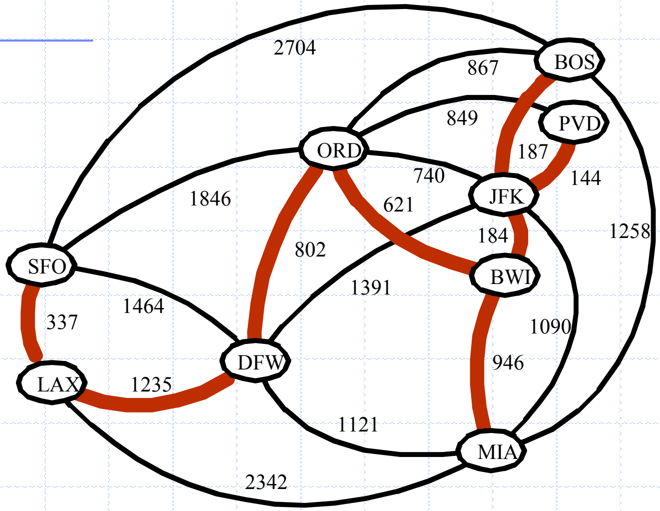
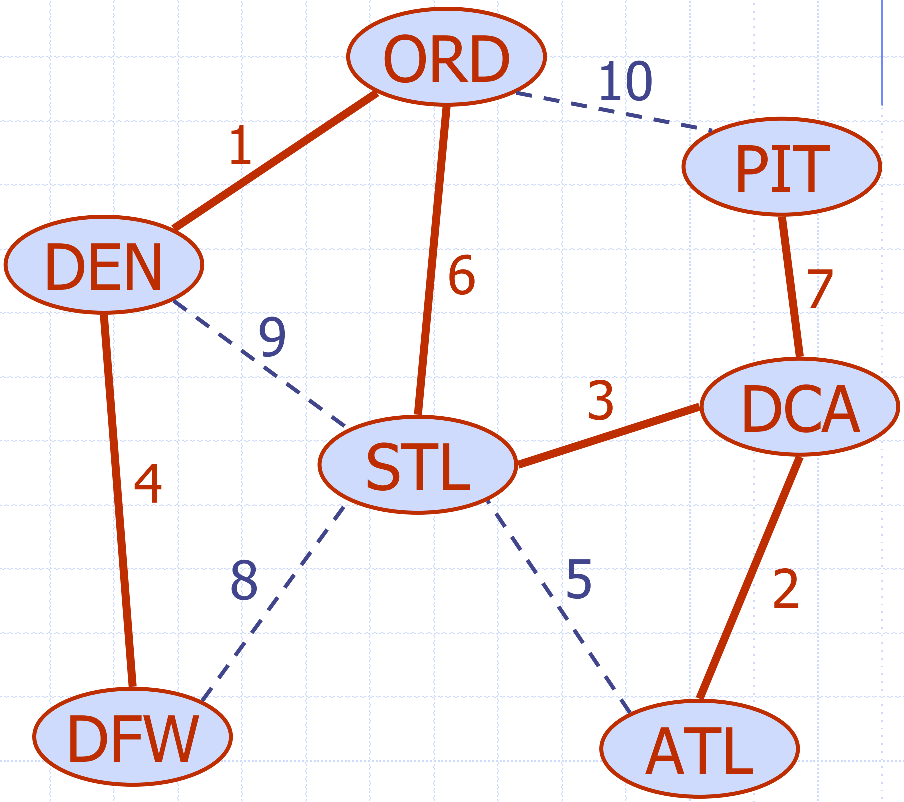
# Minimum Spanning Trees



# Minimum Spanning Trees (§ 12.7)

- Spanning subgraph
  - Subgraph of a graph *G* containing all the vertices of *G*
- Spanning tree
  - Spanning subgraph that is itself a (free) tree
- Minimum spanning tree (MST)
  - Spanning tree of a weighted graph with minimum total edge weight
- Applications
  - Communications networks
  - Transportation networks



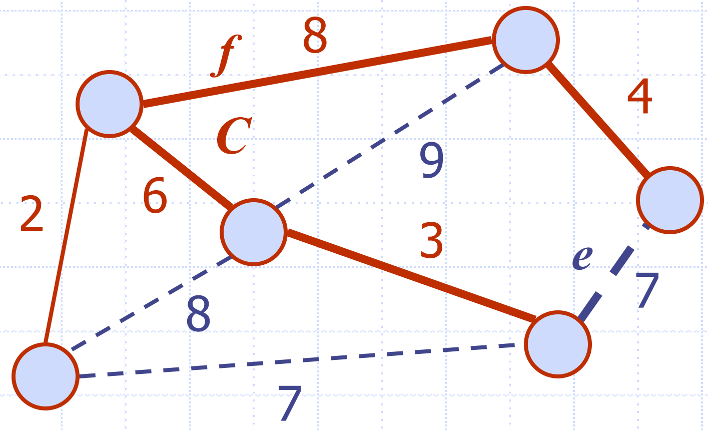
# Cycle Property

Cycle Property:

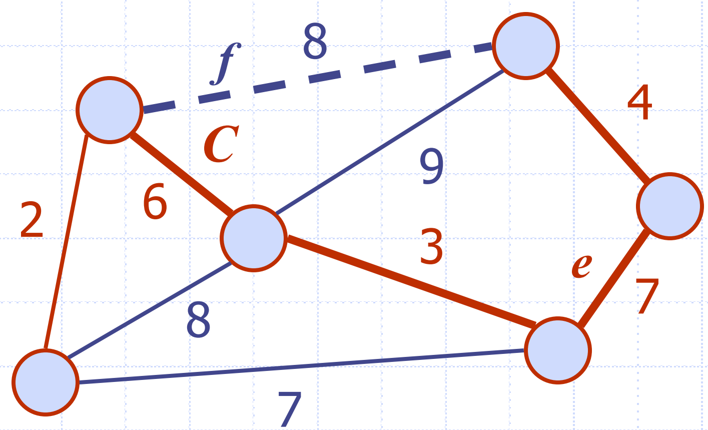
- Let $T$ be a minimum spanning tree of a weighted graph $G$
- Let $e$ be an edge of $G$ that is not in $T$ and $C$ let be the cycle formed by $e$ with $T$
- For every edge $f$ of $C$, $weight(f) \leq weight(e)$

Proof:

- By contradiction
- If $weight(f) > weight(e)$ we can get a spanning tree of smaller weight by replacing $e$ with $f$




Replacing $f$ with $e$ yields a better spanning tree



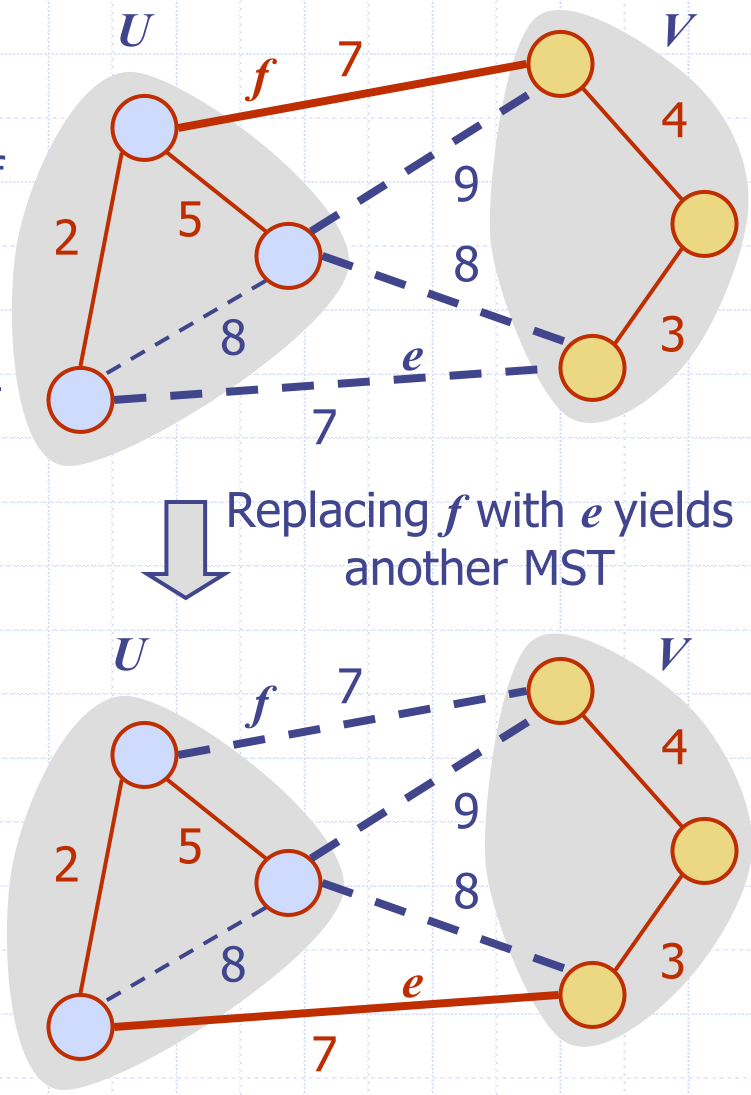
# Partition Property

Partition Property:

- Consider a partition of the vertices of $G$ into subsets $U$ and $V$
- Let $e$ be an edge of minimum weight across the partition
- There is a minimum spanning tree of $G$ containing edge $e$

Proof:

- Let $T$ be an MST of $G$
- If $T$ does not contain $e$, consider the cycle $C$ formed by $e$ with $T$ and let $f$ be an edge of $C$ across the partition
- By the cycle property,
  $$weight(f) \leq weight(e)$$
- Thus, $weight(f) = weight(e)$
- We obtain another MST by replacing $f$ with $e$



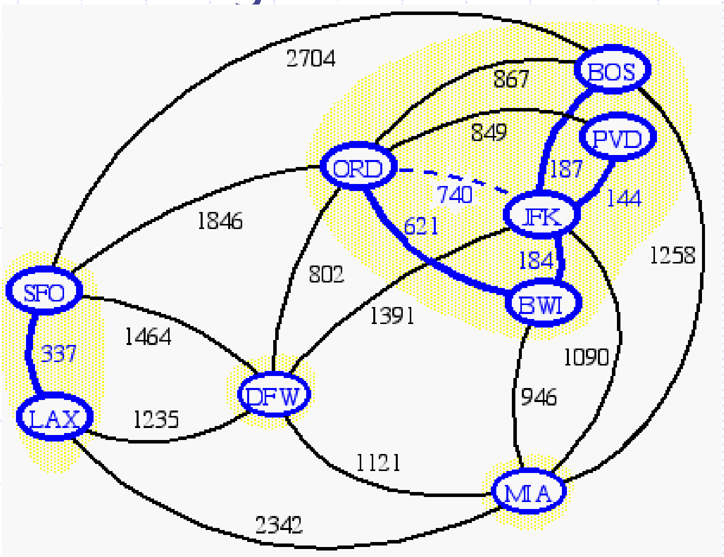
# Kruskal's Algorithm (§ 12.7.1)

- A priority queue stores the edges outside the cloud
  - Key: weight
  - Element: edge
- At the end of the algorithm
  - We are left with one cloud that encompasses the MST
  - A tree *T* which is our MST

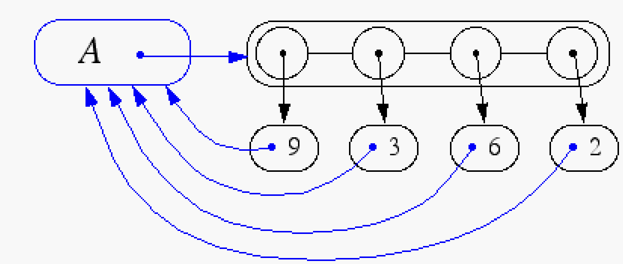
**Algorithm** *KruskalMST(G)*
    **for** each vertex *V* in *G* **do**
        define a ***Cloud(v)*** of $\leftarrow \{v\}$
    **let** *Q* be a priority queue.
    Insert all edges into *Q* using their weights as the key
    $T \leftarrow \varnothing$
    **while** *T* has fewer than *n*-1 edges **do**
        edge ***e*** = ***T.removeMin()***
        Let ***u, v*** be the endpoints of *e*
        **if** ***Cloud(v)*** ≠ ***Cloud(u)*** **then**
            Add edge ***e*** to ***T***
            Merge ***Cloud(v)*** and ***Cloud(u)***
    **return** ***T***

# Data Structure for Kruskal Algortihm (§ 10.6.2)

- The algorithm maintains a forest of trees
- An edge is accepted it if connects distinct trees
- We need a data structure that maintains a **partition**, i.e., a collection of disjoint sets, with the operations:
  - -**find**(u): return the set storing u
  - -**union**(u,v): replace the sets storing u and v with their union



# Representation of a Partition




- Each set is stored in a sequence
- Each element has a reference back to the set
  - operation find(u) takes O(1) time, and returns the set of which u is a member.
  - in operation union(u,v), we move the elements of the smaller set to the sequence of the larger set and update their references
  - the time for operation union(u,v) is $\min(n_u, n_v)$, where $n_u$ and $n_v$ are the sizes of the sets storing u and v
- Whenever an element is processed, it goes into a set of size at least double, hence each element is processed at most log n times

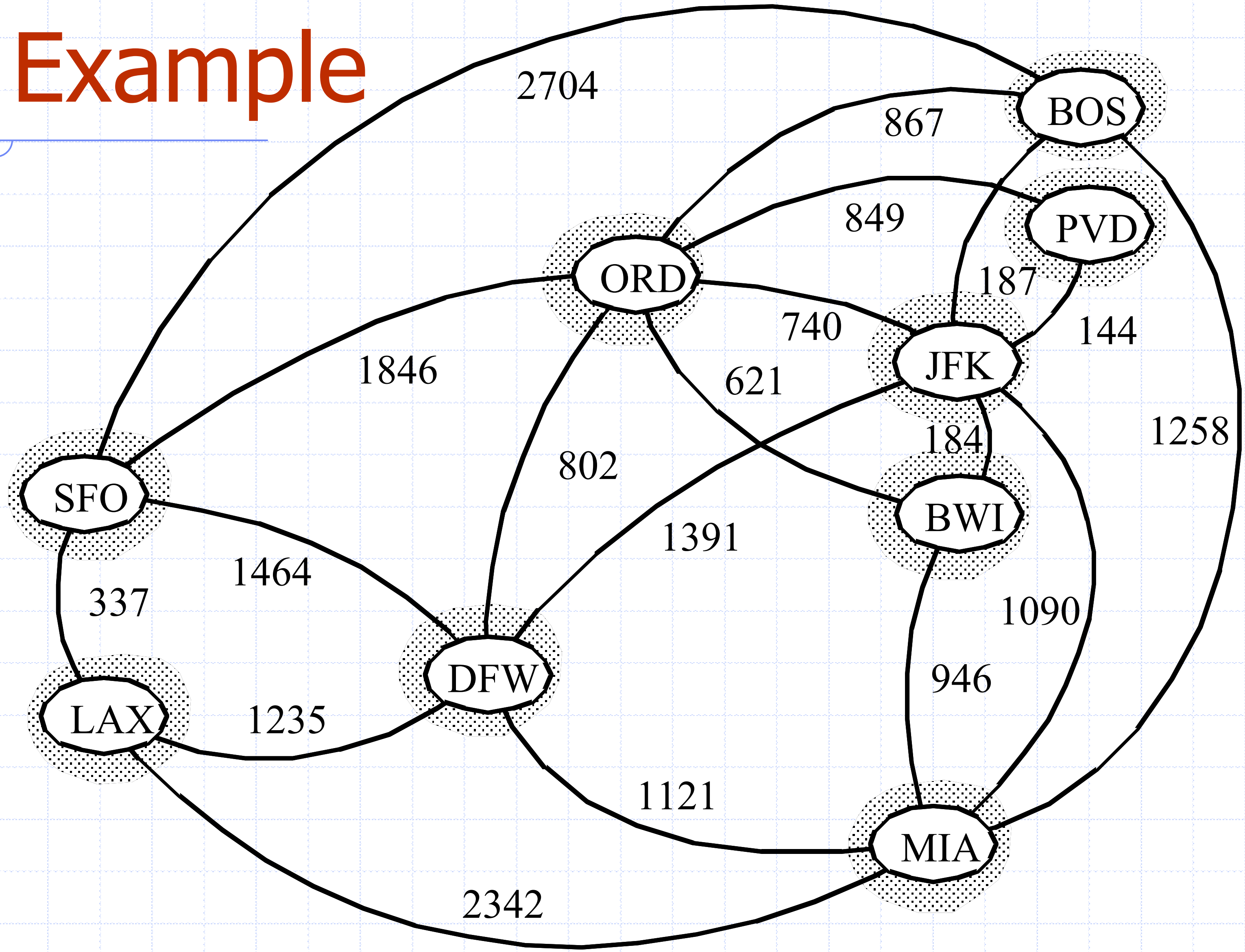
# Partition-Based Implementation

- A partition-based version of Kruskal's Algorithm performs cloud merges as unions and tests as finds.

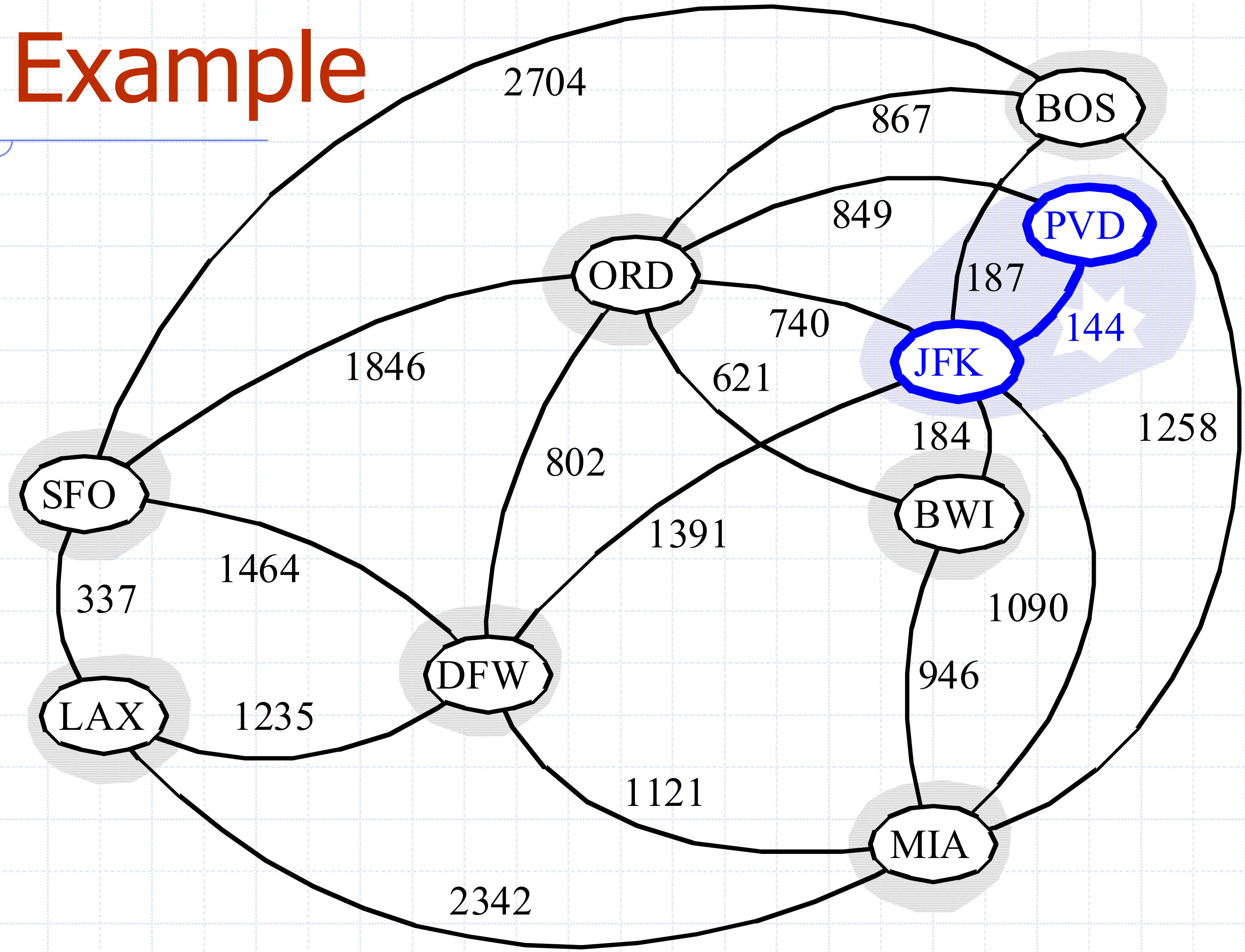
```
Algorithm Kruskal(G):
  Input: A weighted graph G.
  Output: An MST T for G.
Let P be a partition of the vertices of G, where each vertex forms a separate set.
Let Q be a priority queue storing the edges of G, sorted by their weights
Let T be an initially-empty tree
while Q is not empty do
   (u,v) ← Q.removeMinElement()
   if P.find(u) != P.find(v) then
            Add (u,v) to T
            P.union(u,v)
return T
```

Running time: O((n+m)log n)

# Kruskal Example

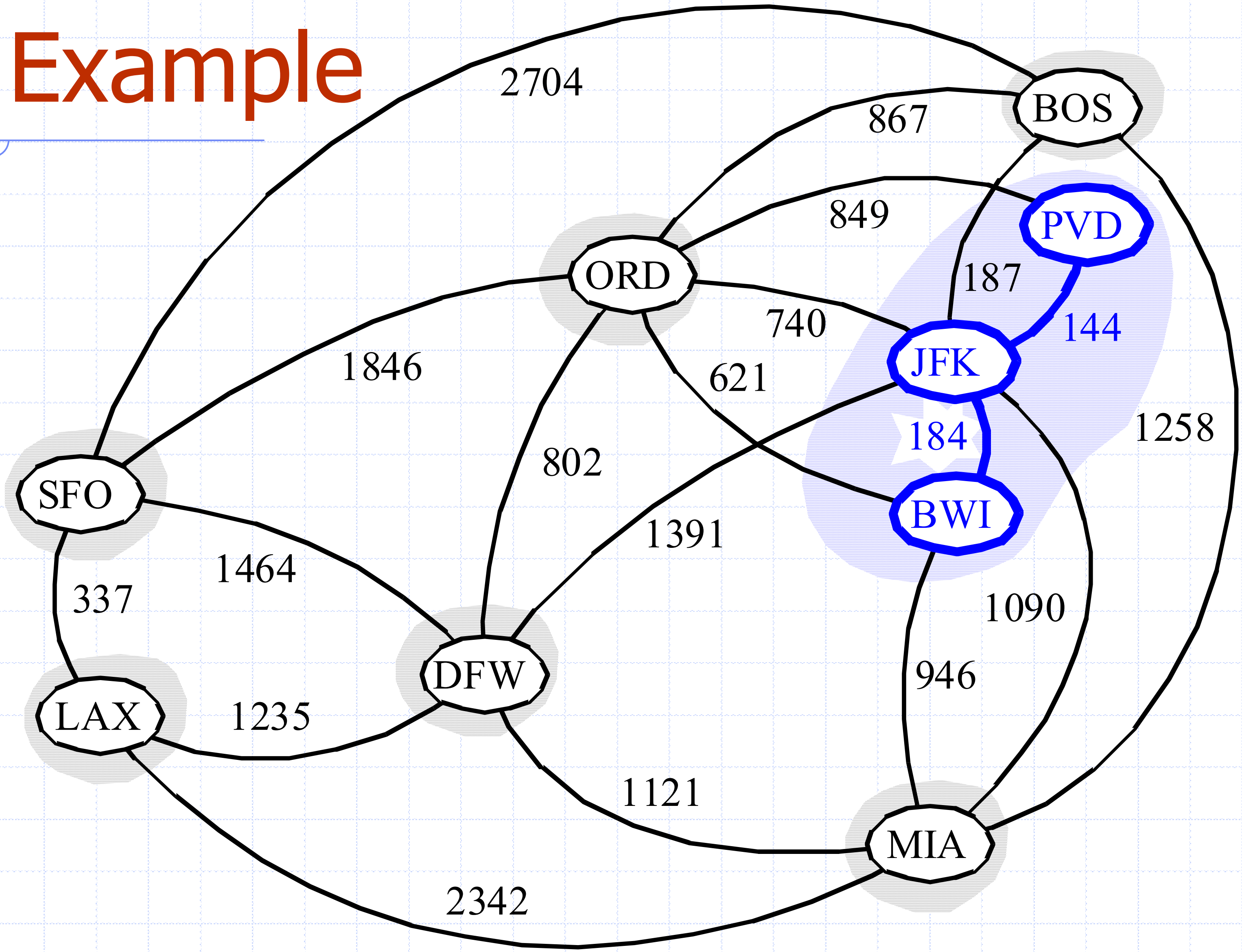


# Example



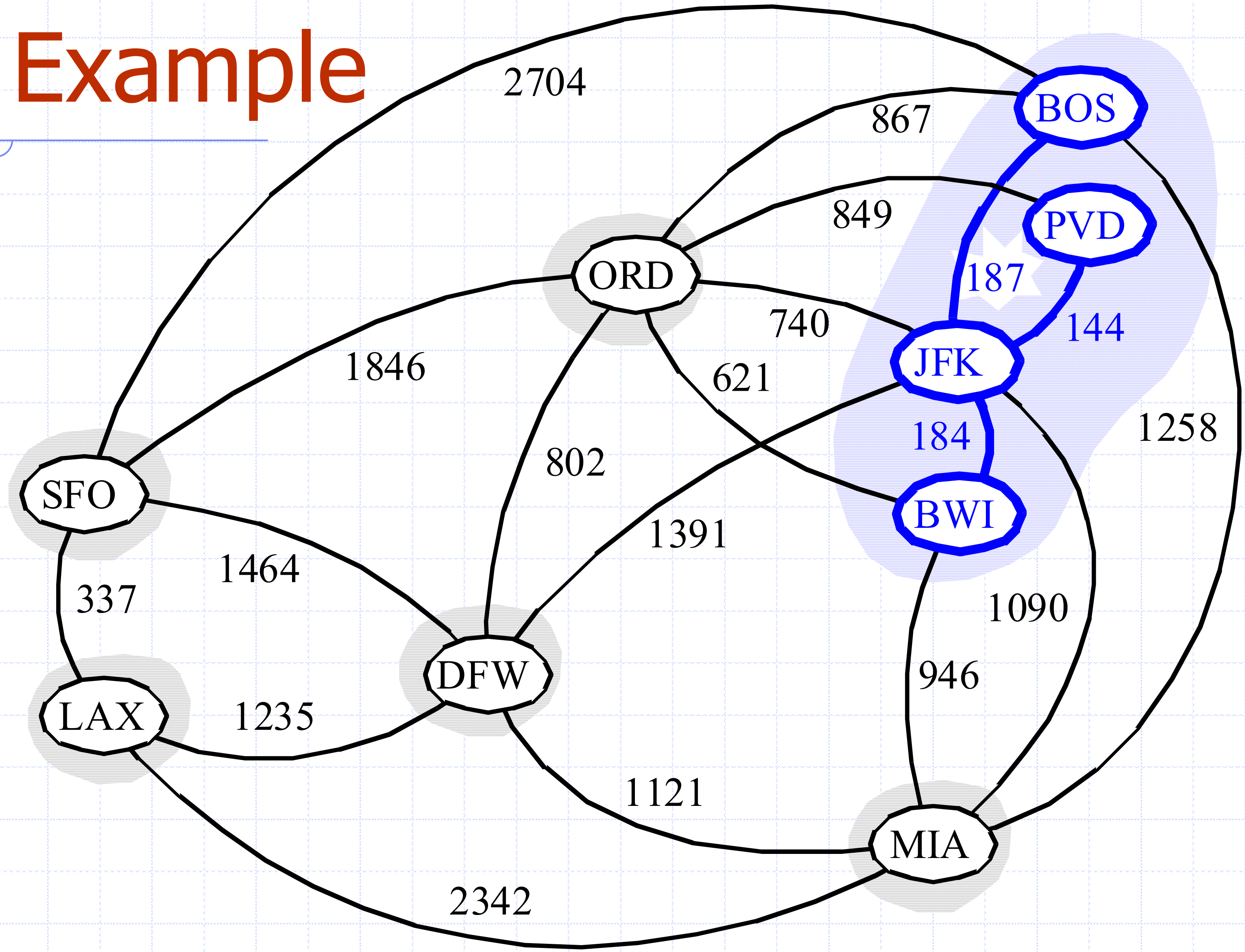
BOS
PVD
ORD
JFK
SFO
BWI
DFW
LAX
MIA
2704
867
849
187
144
740
621
1846
802
184
1258
1391
1464
337
1090
946
1235
1121
2342

# Example



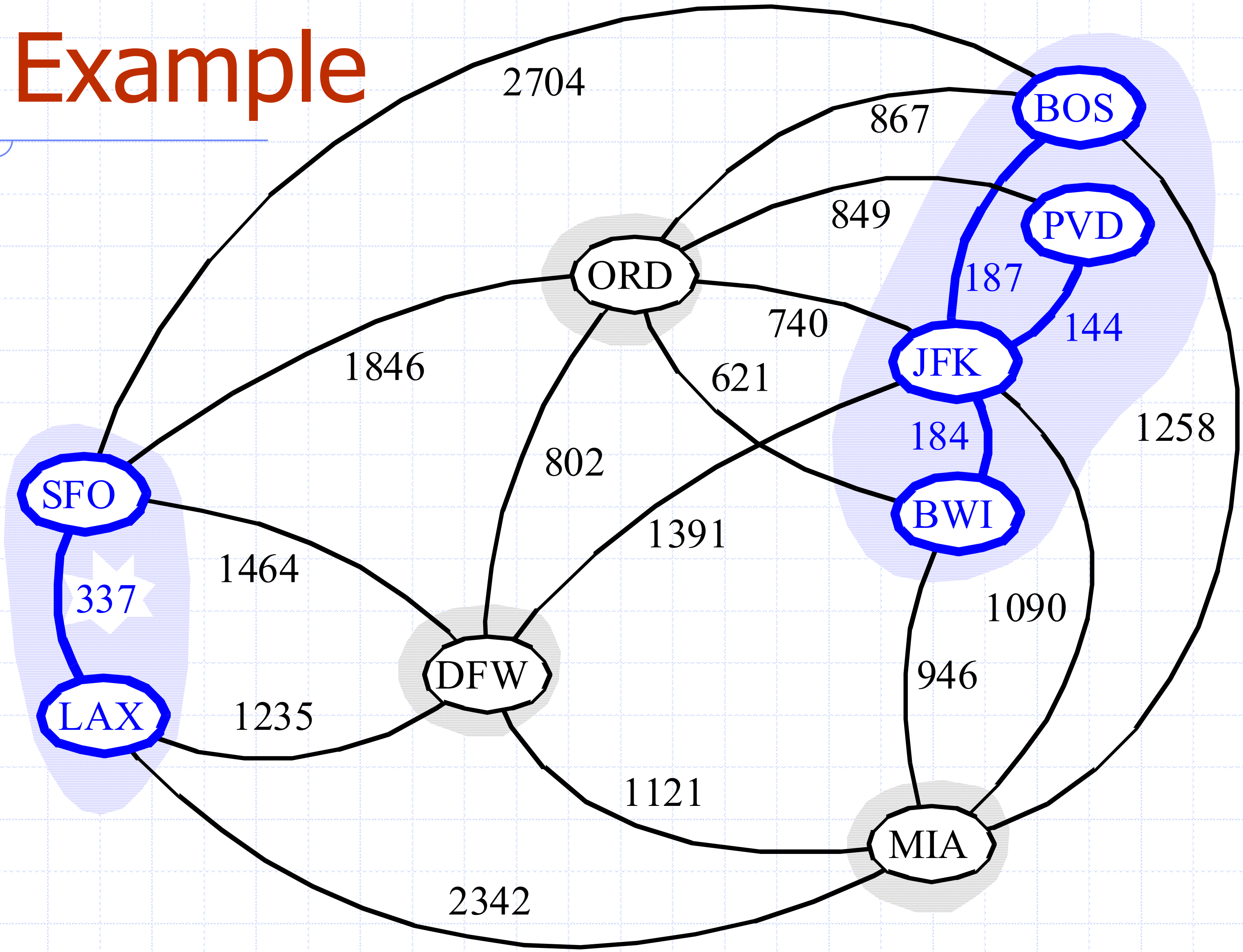
2704
BOS
867
849
PVD
ORD
187
740
144
1846
621
JFK
1258
SFO
802
184
1391
BWI
1464
337
1090
DFW
946
LAX
1235
1121
MIA
2342

# Example



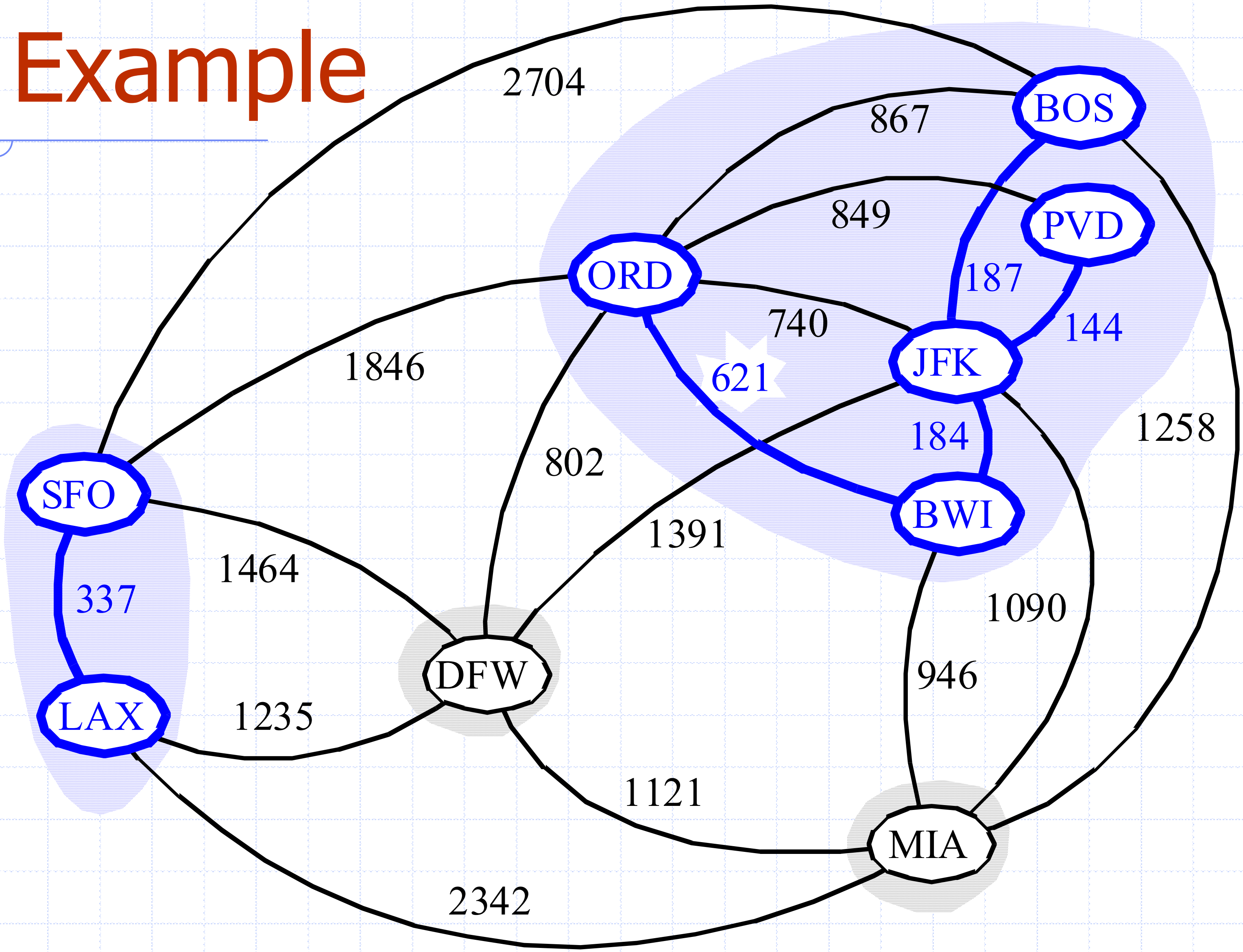
2704
BOS
867
849
PVD
ORD
187
740
1846
621
JFK
144
1258
184
802
SFO
BWI
1391
1464
337
1090
DFW
946
LAX
1235
1121
MIA
2342

# Example



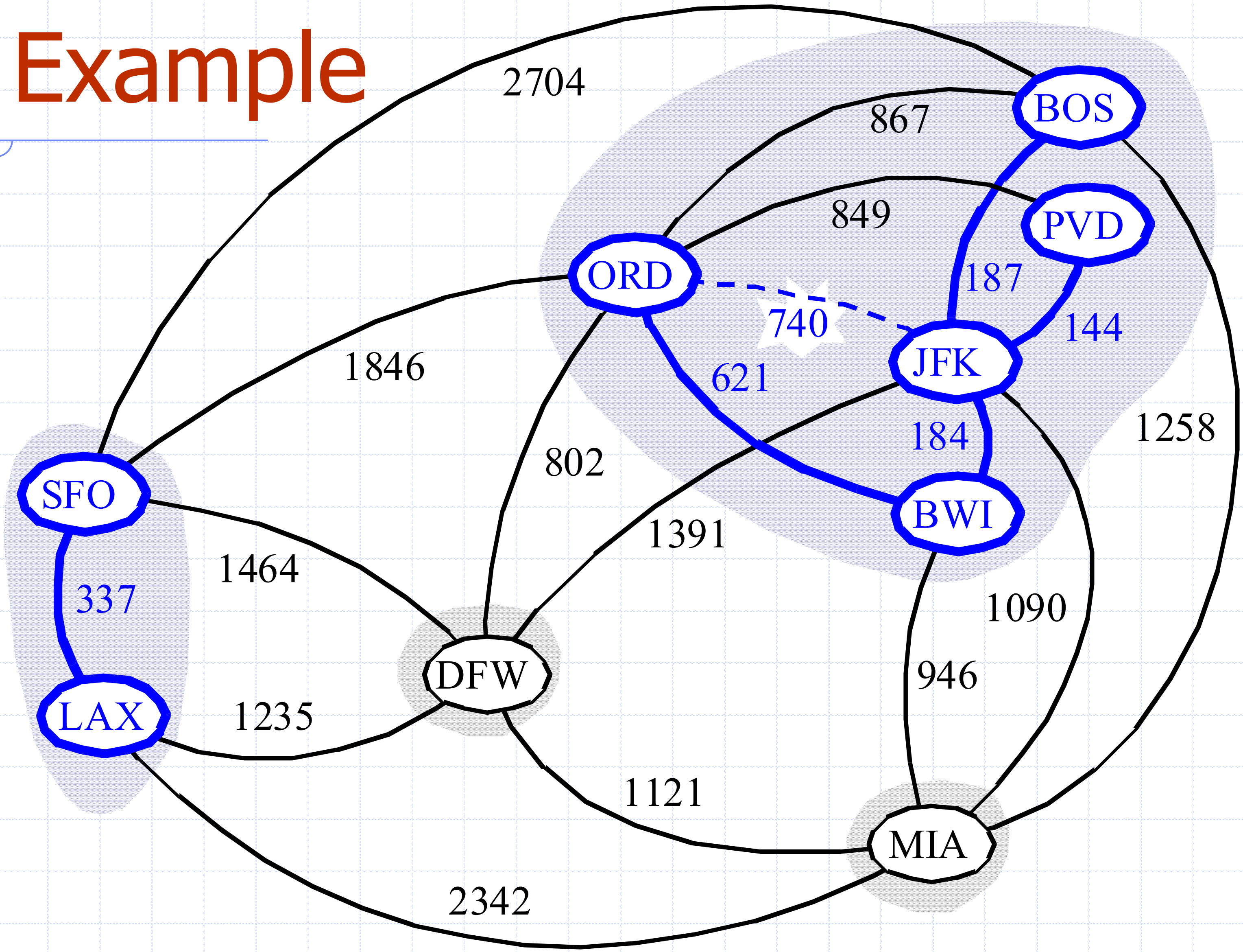
2704
867
BOS
849
PVD
ORD
187
740
144
1846
621
JFK
1258
184
802
SFO
BWI
1391
1464
337
1090
DFW
946
LAX
1235
1121
MIA
2342

# Example



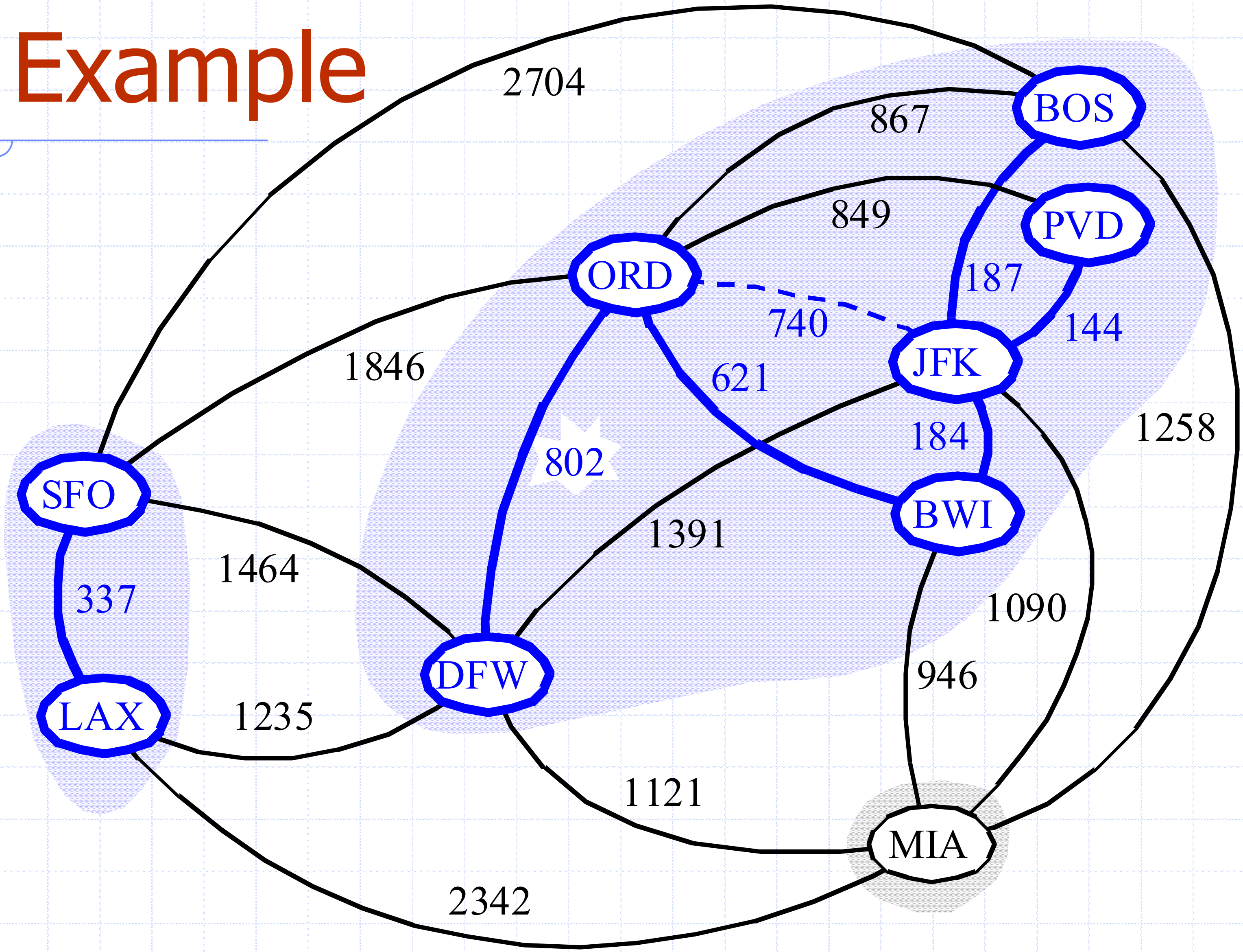
2704
BOS
867
849
PVD
ORD
187
144
740
1846
621
JFK
1258
184
802
SFO
BWI
1391
1464
337
1090
DFW
946
LAX
1235
1121
MIA
2342

# Example



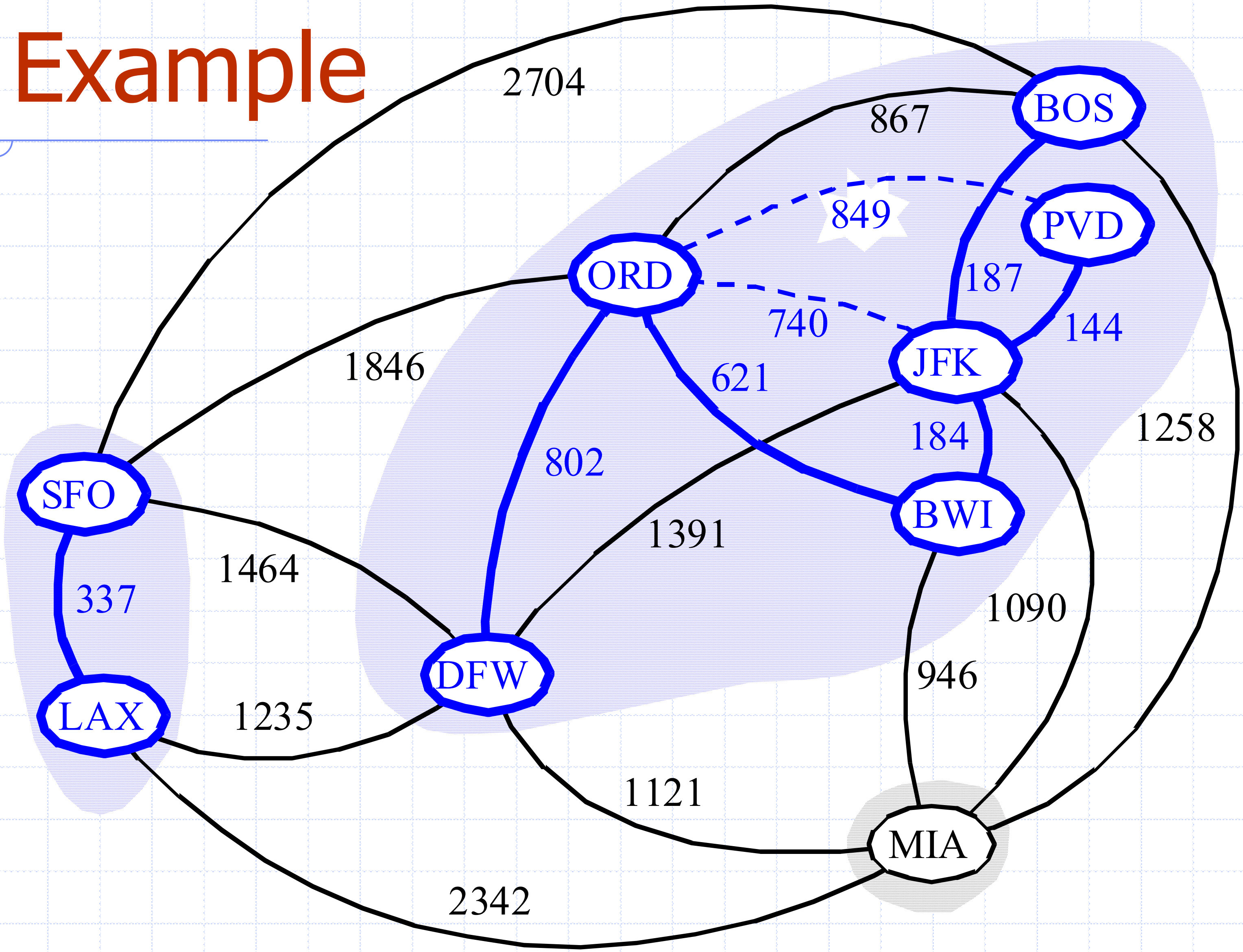
2704
867
BOS
849
PVD
ORD
187
740
144
JFK
1846
621
184
1258
802
SFO
BWI
1391
1464
337
1090
DFW
946
LAX
1235
1121
MIA
2342

# Example



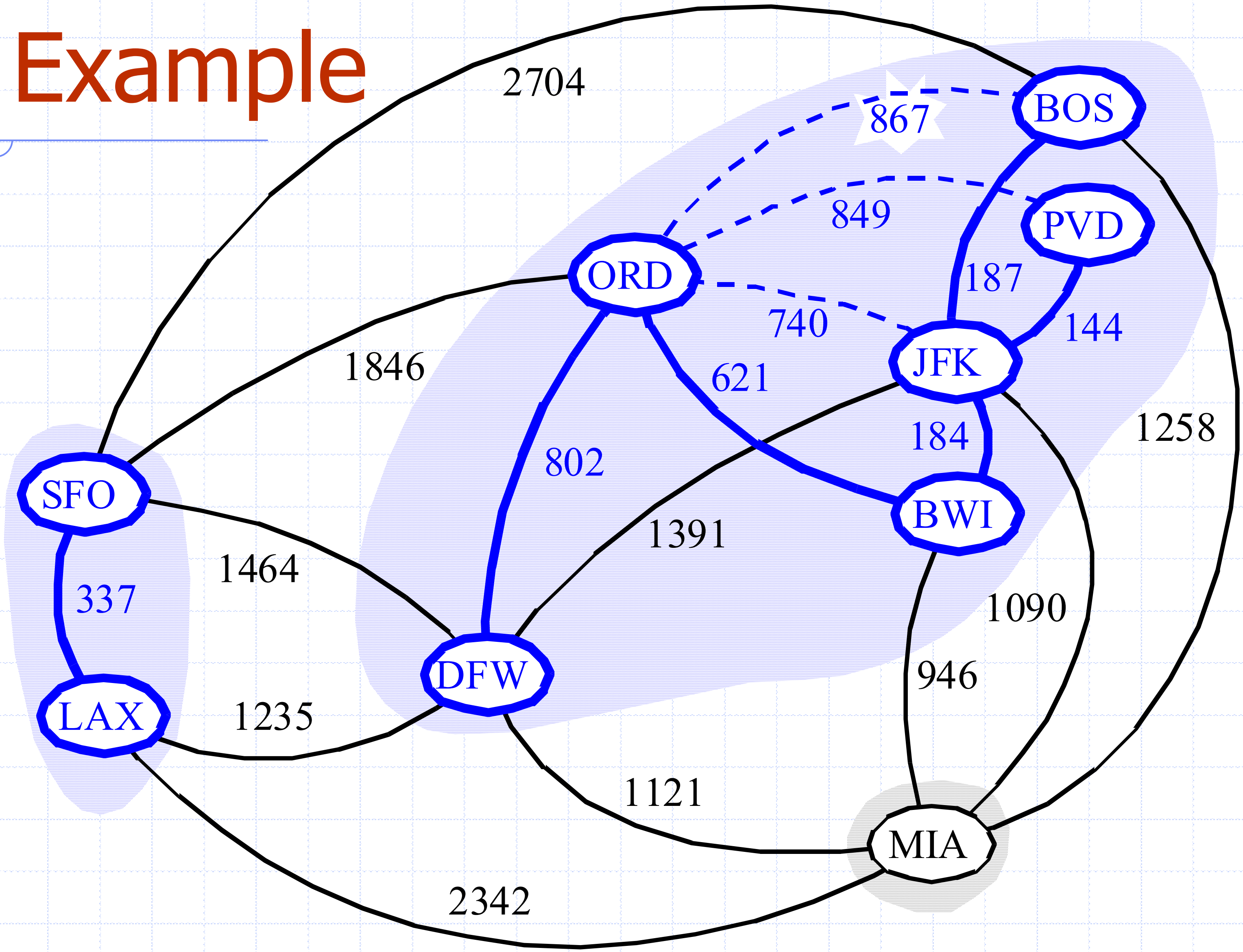
2704
BOS
867
PVD
849
ORD
187
740
144
JFK
1846
621
1258
184
802
SFO
BWI
1391
1464
337
1090
DFW
946
LAX
1235
1121
MIA
2342

# Example



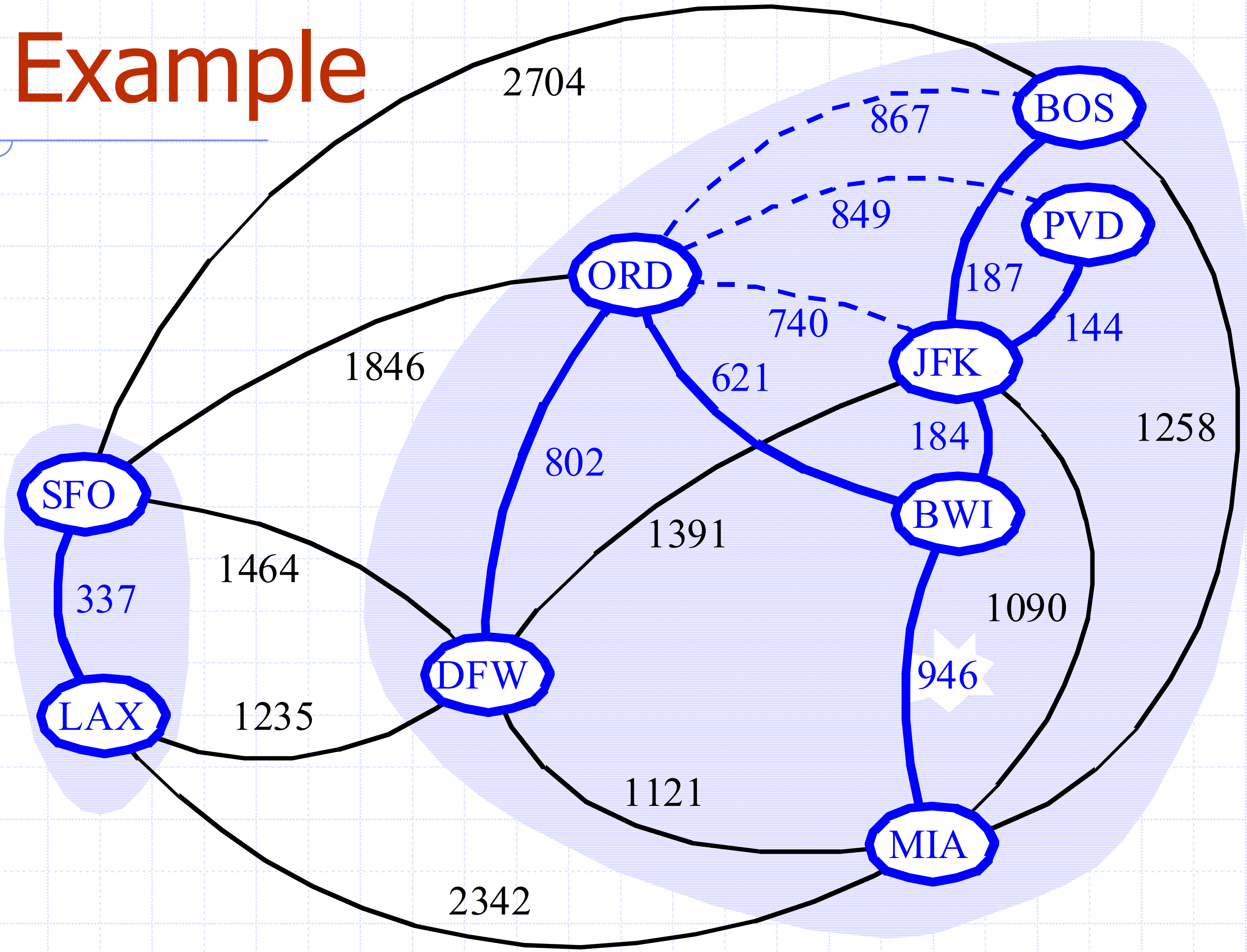
2704
BOS
867
849
PVD
ORD
187
740
144
1846
621
JFK
1258
184
802
SFO
BWI
1391
1464
337
1090
DFW
946
LAX
1235
1121
MIA
2342

# Example



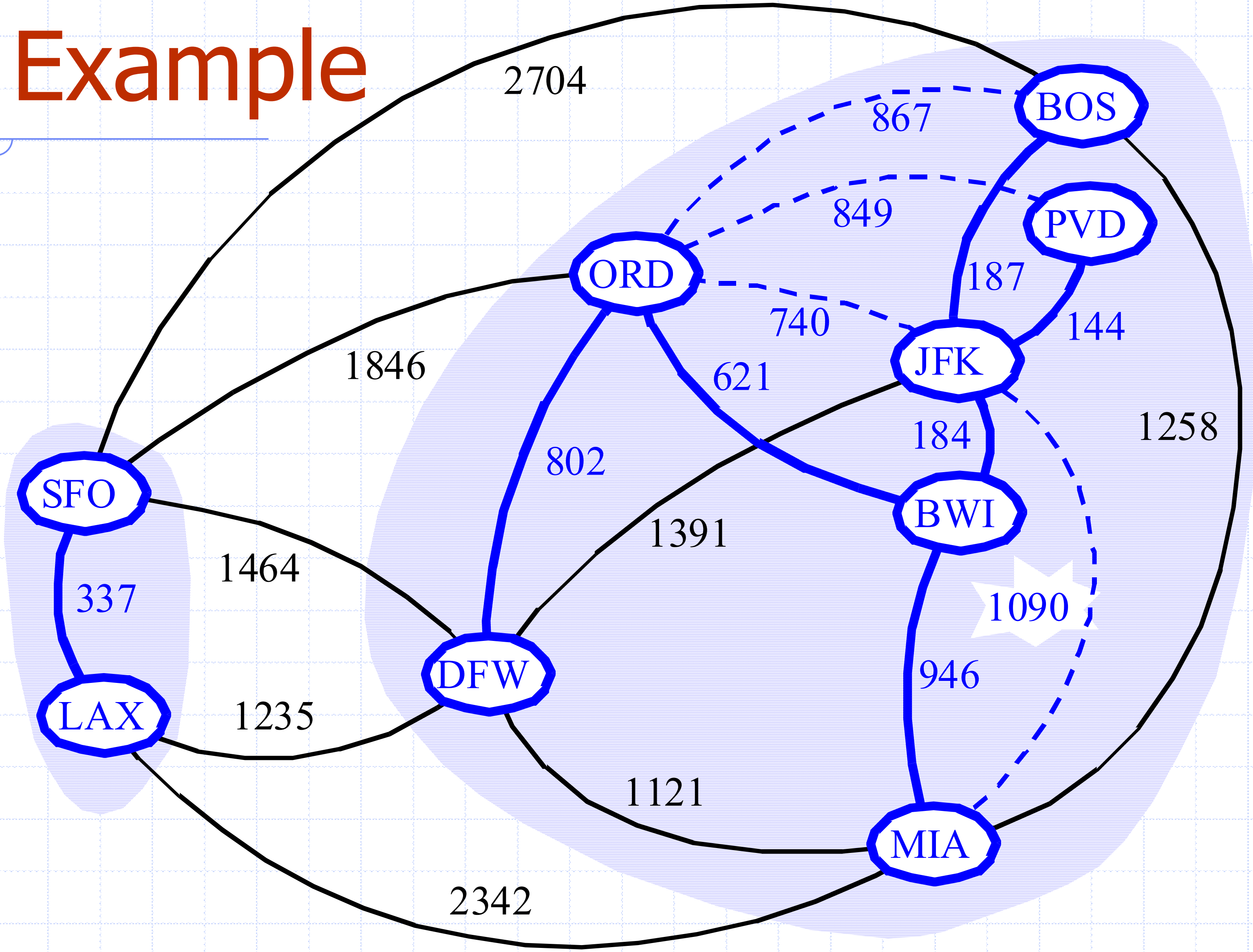
2704
BOS
867
849
PVD
ORD
187
740
144
1846
621
JFK
184
1258
802
SFO
BWI
1391
1464
337
1090
DFW
946
LAX
1235
1121
MIA
2342

# Example



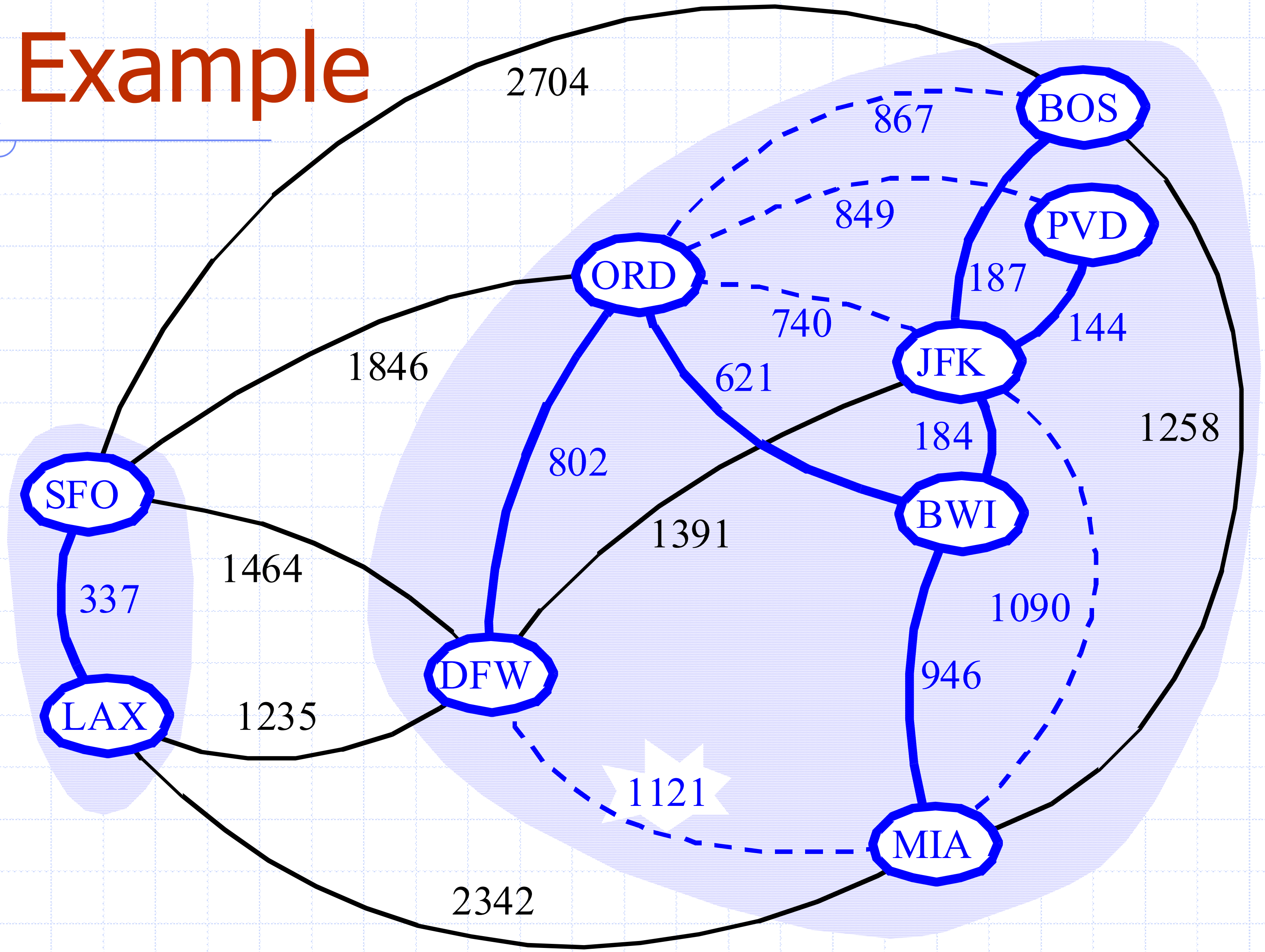
2704
BOS
867
PVD
849
ORD
187
740
144
1846
621
JFK
184
1258
802
SFO
BWI
1391
1464
337
1090
DFW
946
LAX
1235
1121
MIA
2342

# Example



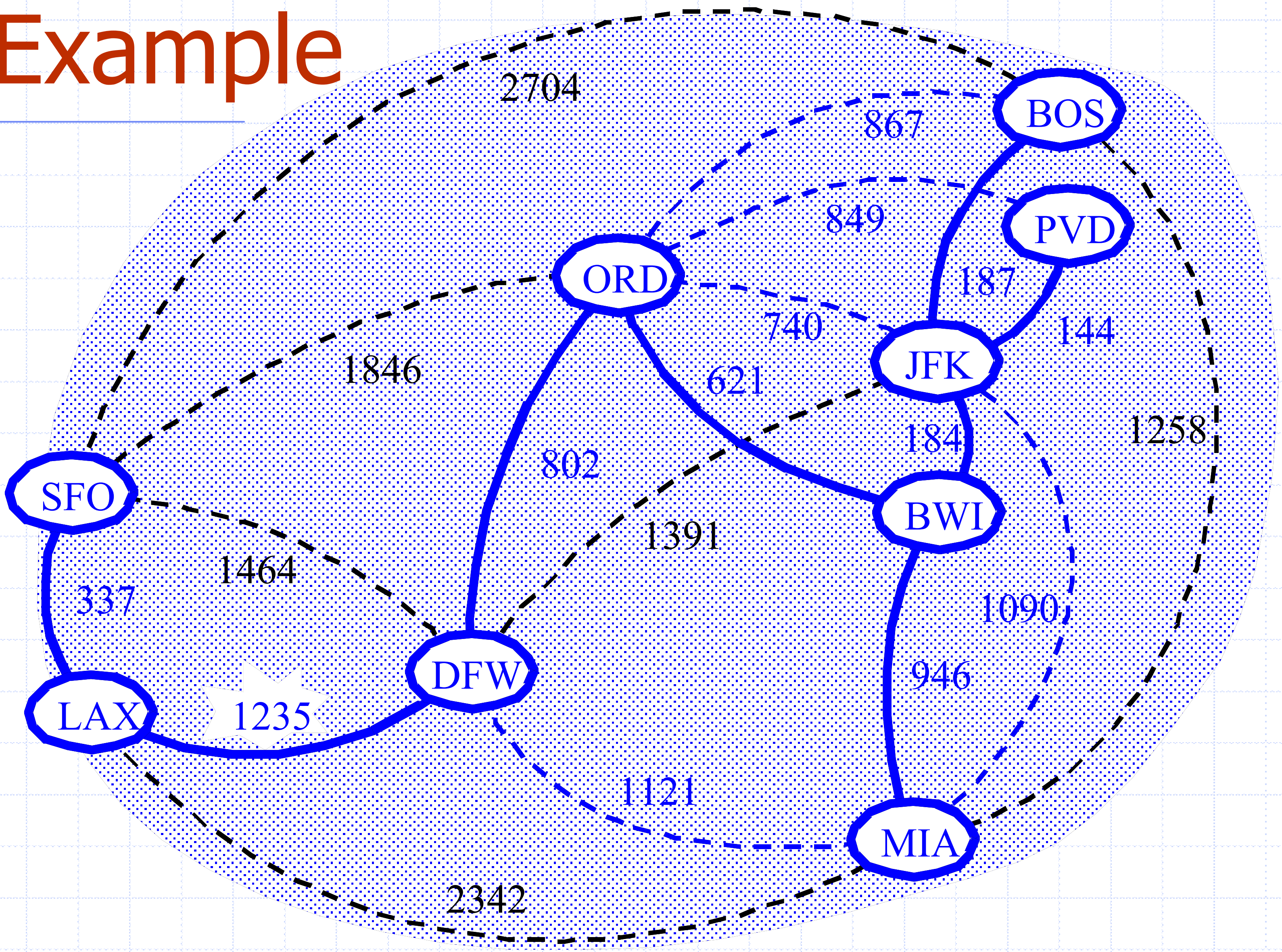
2704
867
BOS
849
PVD
ORD
187
740
144
1846
621
JFK
1258
184
802
SFO
BWI
1391
1464
1090
337
946
DFW
LAX
1235
1121
MIA
2342

# Example



2704
867
BOS
849
PVD
ORD
187
740
144
1846
JFK
621
1258
184
802
SFO
BWI
1391
1464
337
1090
DFW
946
LAX
1235
1121
MIA
2342

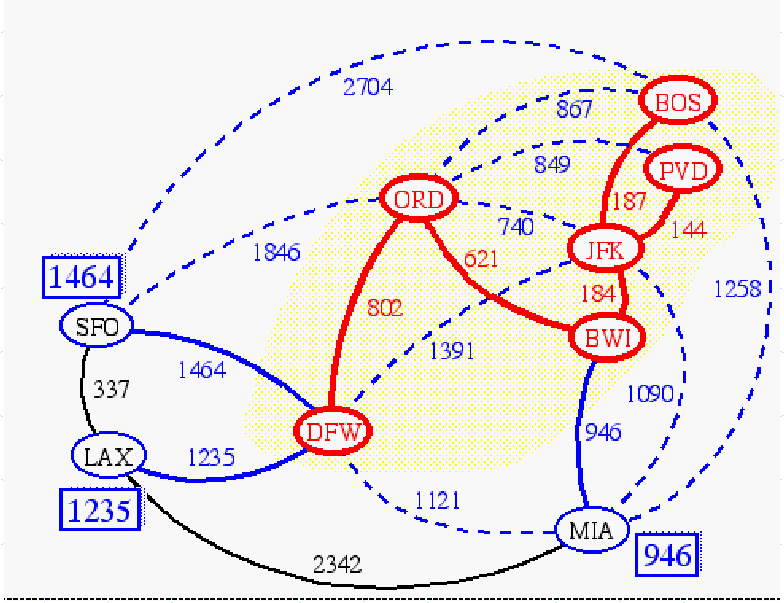
# Example



2704
867
BOS
849
PVD
ORD
187
740
144
1846
621
JFK
1258
184
802
SFO
BWI
1391
1464
337
1090
DFW
946
LAX
1235
1121
MIA
2342

# Prim-Jarnik's Algorithm (§ 12.7.2)

- Similar to Dijkstra's algorithm (for a connected graph)
- We pick an arbitrary vertex $s$ and we grow the MST as a cloud of vertices, starting from $s$
- We store with each vertex $v$ a label $d(v)$ = the smallest weight of an edge connecting $v$ to a vertex in the cloud
- At each step:
  - We add to the cloud the vertex $u$ outside the cloud with the smallest distance label
  - We update the labels of the vertices adjacent to $u$

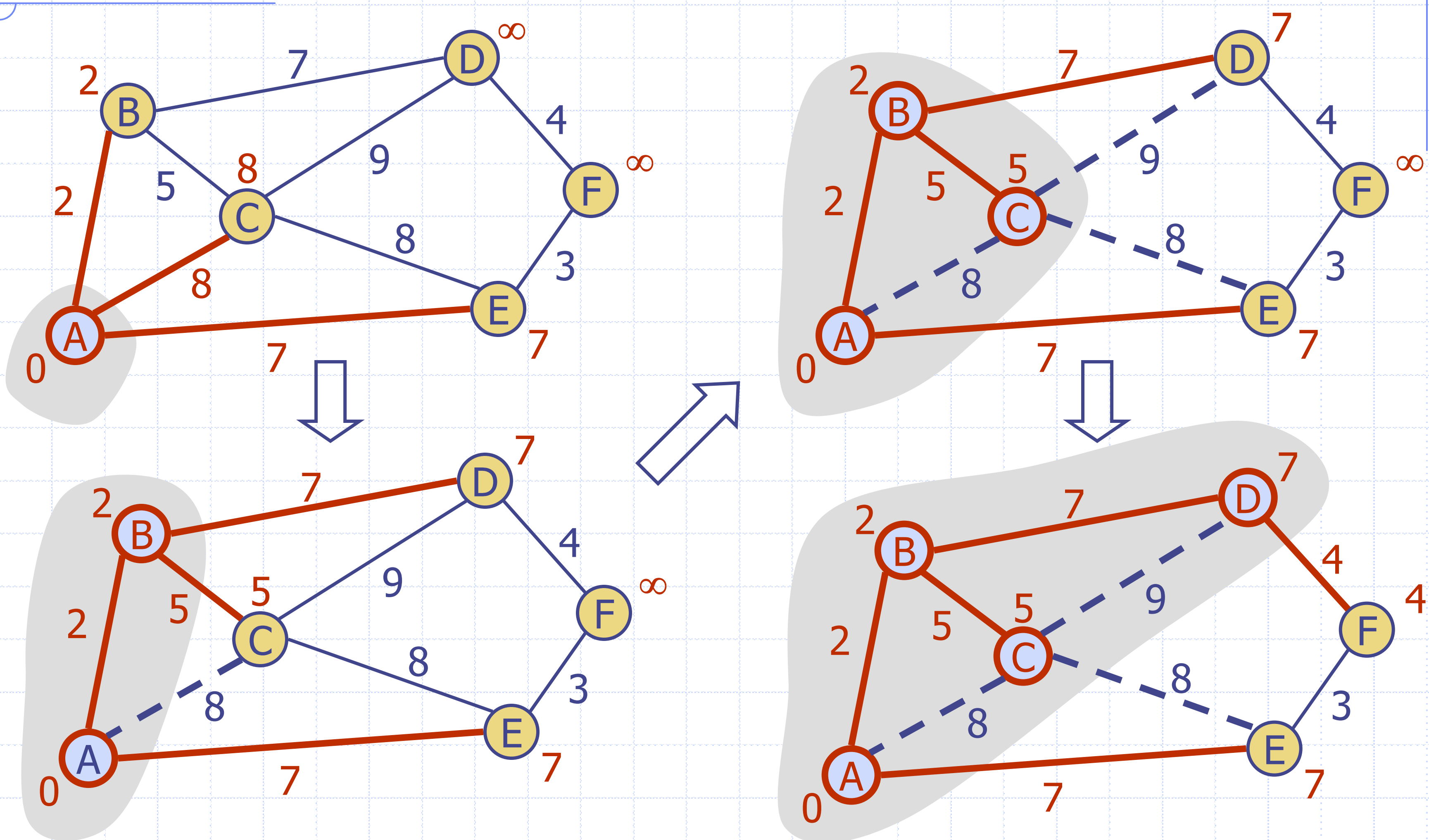


# Prim-Jarnik's Algorithm (cont.)

- A priority queue stores the vertices outside the cloud
  - Key: distance
  - Element: vertex
- Locator-based methods
  - *insert*(*k*,*e*) returns a locator
  - *replaceKey*(*l*,*k*) changes the key of an item
- We store three labels with each vertex:
  - Distance
  - Parent edge in MST
  - Locator in priority queue

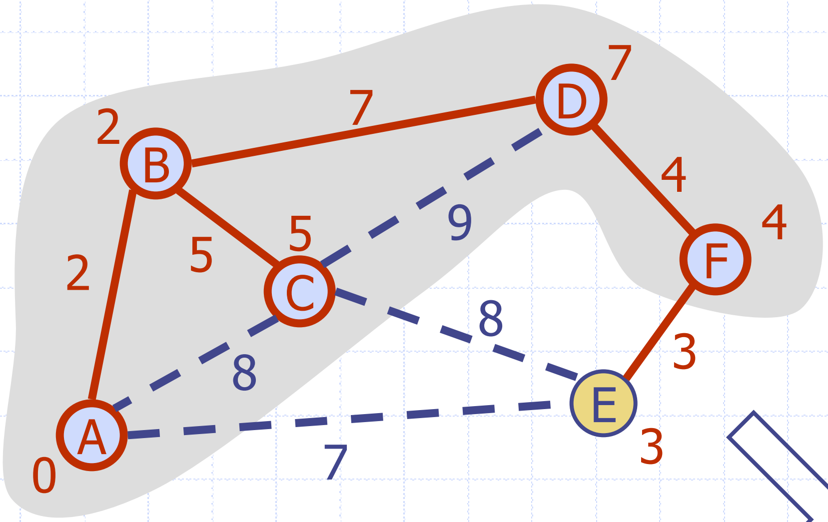
```
Algorithm PrimJarnikMST(G)
    Q ← new heap-based priority queue
    s ← a vertex of G
    for all v ∈ G.vertices()
        if v = s
            setDistance(v, 0)
        else
            setDistance(v, ∞)
        setParent(v, ∅)
        l ← Q.insert(getDistance(v), v)
        setLocator(v,l)
    while ¬Q.isEmpty()
        u ← Q.removeMin()
        for all e ∈ G.incidentEdges(u)
            z ← G.opposite(u,e)
            r ← weight(e)
            if r < getDistance(z)
                setDistance(z,r)
                setParent(z,e)
                Q.replaceKey(getLocator(z),r)
```

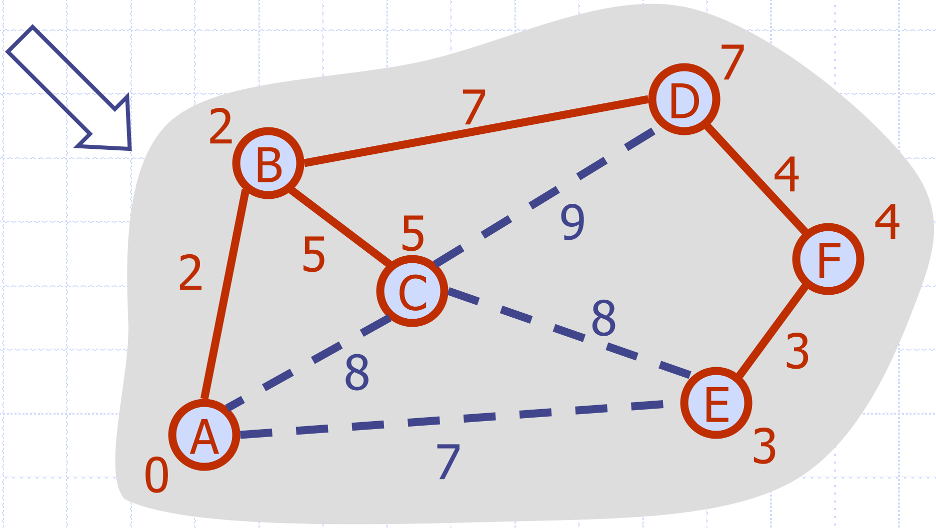
# Example



2 B 7 D ∞
4
2 5 8 9 F ∞
C
8 3
8
A E
0 7 7
2 B 7 D 7
4
2 5 5 9 F ∞
C
8 3
8
A E
0 7 7
2 B 7 D 7
4
2 5 5 9 F ∞
C
8 3
8
A E
0 7 7
2 B 7 D 7
4
2 5 5 9 F 4
C
8 3
8
A E
0 7 7

# Example (contd.)



# Analysis

- Graph operations
  - Method incidentEdges is called once for each vertex
- Label operations
  - We set/get the distance, parent and locator labels of vertex $z$ $O(\deg(z))$ times
  - Setting/getting a label takes $O(1)$ time
- Priority queue operations
  - Each vertex is inserted once into and removed once from the priority queue, where each insertion or removal takes $O(\log n)$ time
  - The key of a vertex $w$ in the priority queue is modified at most $\deg(w)$ times, where each key change takes $O(\log n)$ time
- Prim-Jarnik's algorithm runs in $O((n + m) \log n)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\Sigma_v \deg(v) = 2m$
- The running time is $O(m \log n)$ since the graph is connected

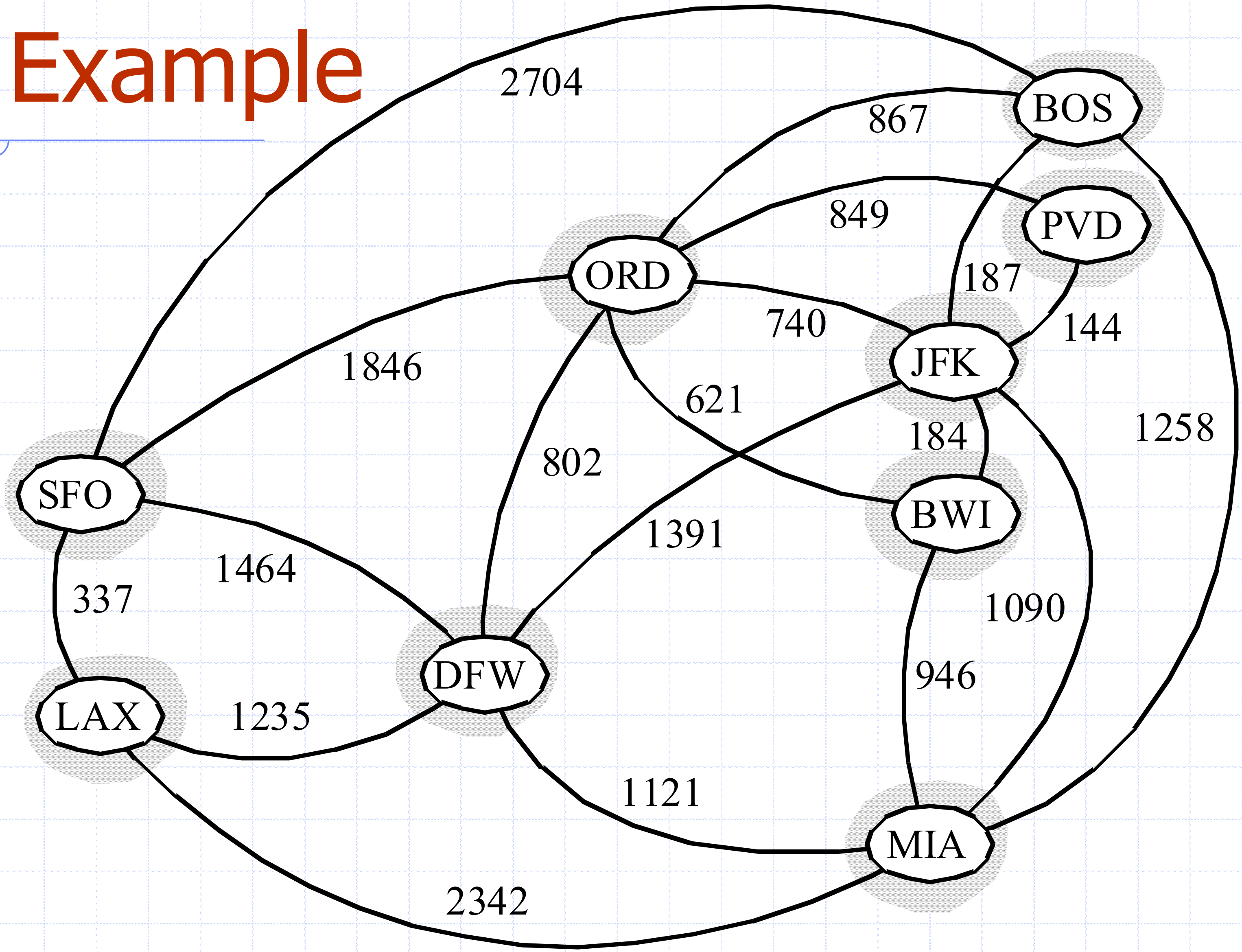
# Baruvka's Algorithm (Ex. C-12.28)

- Like Kruskal's Algorithm, Baruvka's algorithm grows many "clouds" at once.

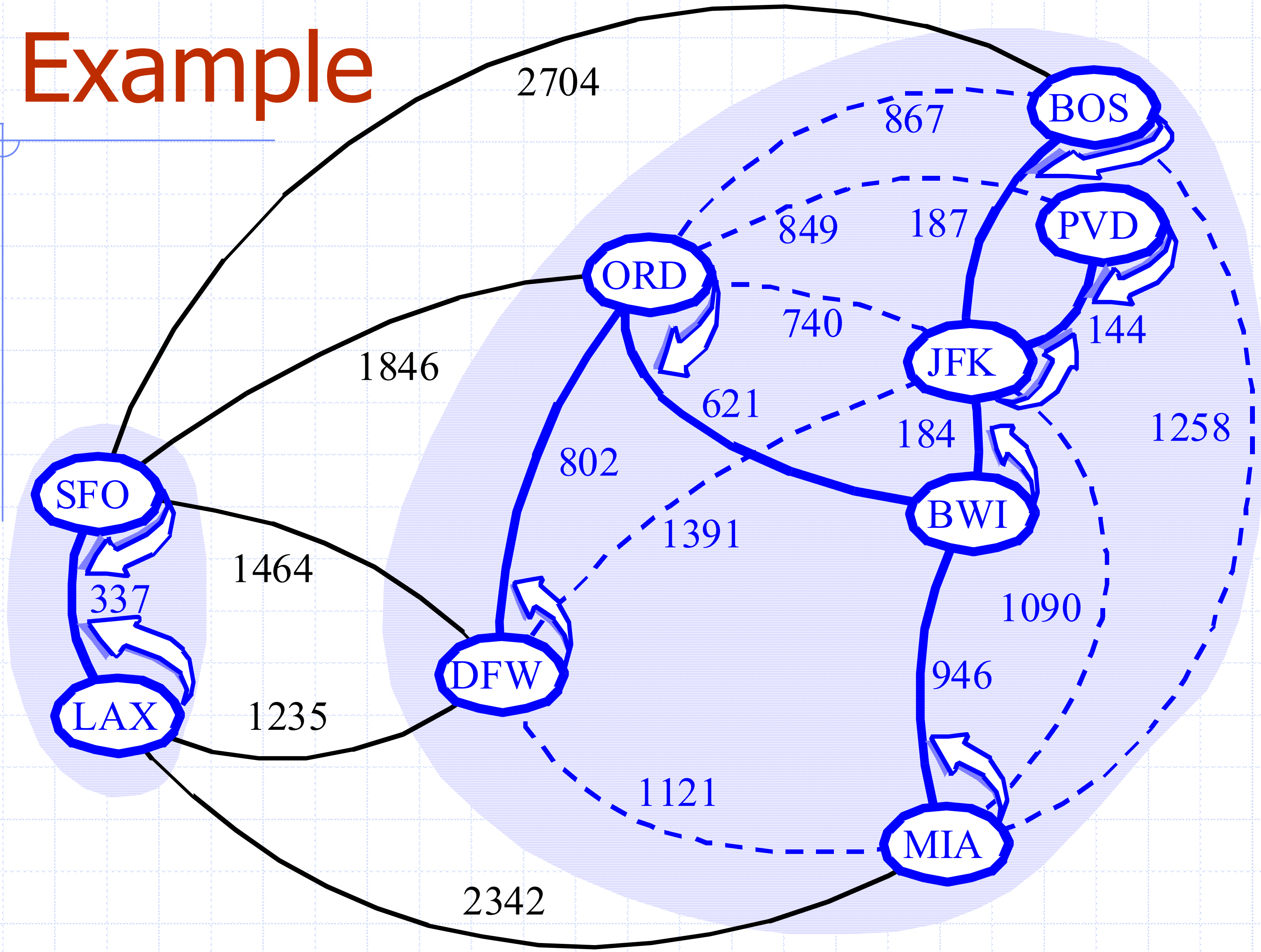
**Algorithm** *BaruvkaMST*(*G*)
    *T* ← *V* {just the vertices of *G*}
    **while** *T* has fewer than ***n***-1 edges **do**
        **for each** connected component *C* in *T* **do**
            Let edge ***e*** be the smallest-weight edge from *C* to another component in ***T***.
            **if** ***e*** is not already in *T* **then**
                Add edge ***e*** to *T*
    **return** *T*

- Each iteration of the while-loop halves the number of connected compontents in T.
  - The running time is O(m log n).

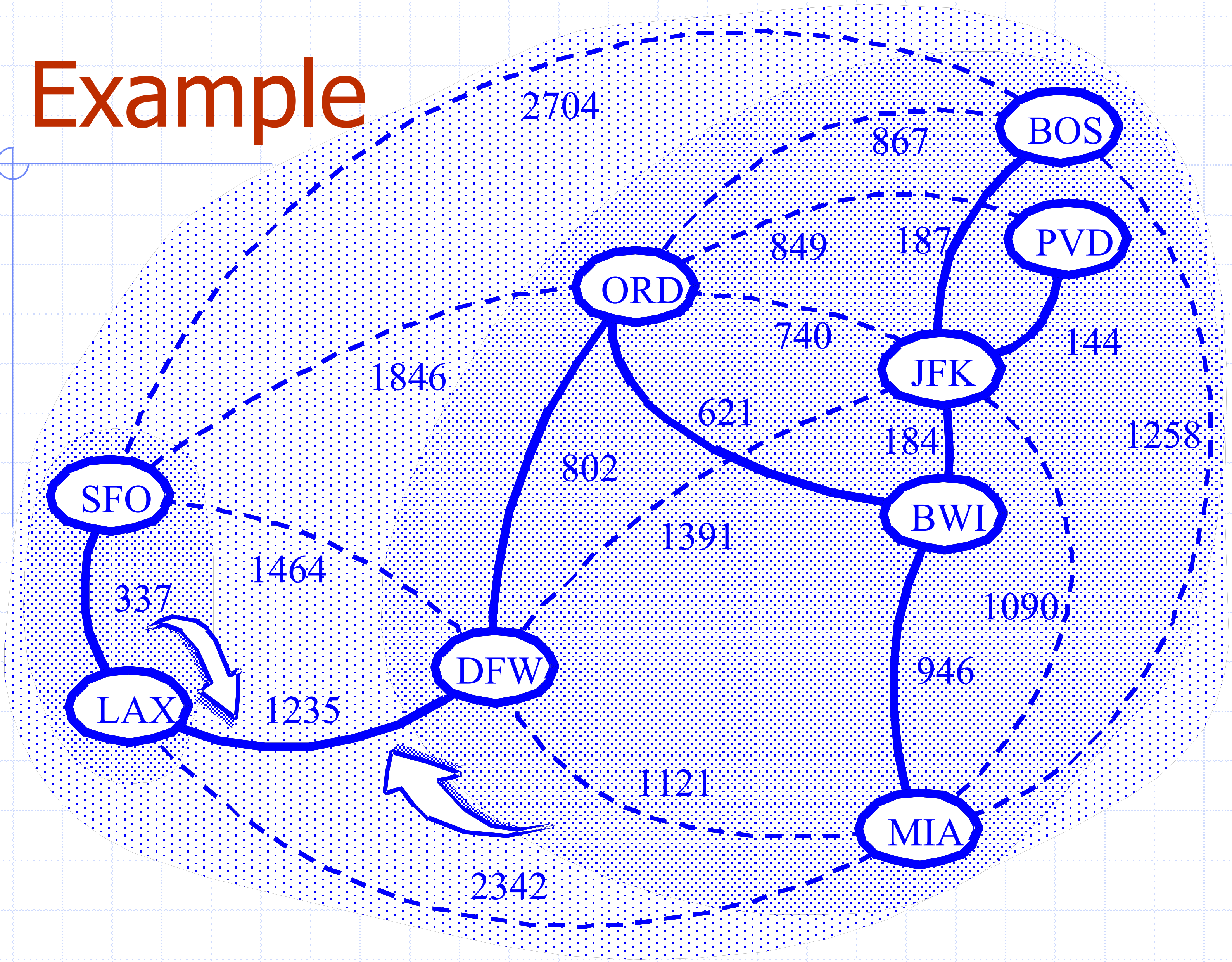
# Baruvka Example



# Example


2704
867
BOS
849
187
PVD
ORD
740
144
1846
JFK
621
184
1258
802
SFO
BWI
1391
1464
337
1090
DFW
946
LAX
1235
1121
MIA
2342

# Example



2704
867
BOS
849
187
PVD
ORD
740
144
JFK
1846
621
184
1258
802
SFO
BWI
1391
1464
337
1090
DFW
946
LAX
1235
1121
MIA
2342