

60-100 Lecture 23

PROOF THEORY

Topics

1. Formal deduction system
2. Proofs
3. Soundness and completeness
4. Theories
5. Consistency
6. Resolution theorem proving

Formal deduction system

Is a syntactic method for generating new formulae from given formulae.

e.g.

An **axiom system** consists of:

logical axioms (e.g. $\mathbf{a} \vee \neg \mathbf{a}$)

inference rules

(e.g. from \mathbf{a} and $\mathbf{a} \rightarrow \mathbf{b}$, infer \mathbf{b})

Note that the logical axioms are true in all truth assignments. They are sometimes called **tautologies**.

Note that the inference rules can be used with consistent substitution.

Formal Proof

A formal proof of F from a set of statements S in a formal deduction system A is a finite sequence F_1, F_2, \dots, F_n such that

$$F = F_n$$

and for each F_i , $1 \leq i \leq n$

Either

1. F_i is a logical axiom of A
2. or $F_i \in S$
3. or F_i is generated from previous members of the sequence using the inference rules of A .

The existence of a proof of F from S in a system A is denoted by:

$$S \vdash_A F$$

An example proof

Axiom system A: *logical axioms*

$$= \{ \mathbf{a1: P \vee \neg P}, \quad \mathbf{a2: \neg \neg P \rightarrow P}, \\ \mathbf{a3: (P \wedge Q) \rightarrow P}, \mathbf{a4: (P \wedge Q) \rightarrow Q} \}$$

inference rules =

{r1 = from **a** and **a** \rightarrow **b**, infer **b**,

r2 = if **F** is a logical axiom, then so is **F'**,
where **F'** is obtained from **F** by
consistent substitution}

$$\mathbf{S = \{s1: pat\ is\ a\ student \wedge pat\ is\ a\ girl, \\ s2: pat\ is\ a\ girl \rightarrow pat\ is\ human\}}$$

Show that $\mathbf{S \vdash_A pat\ is\ human}$

Proof

(i) from a4 and r2, infer:

$$\mathbf{pat\ is\ a\ student \wedge pat\ is\ a\ girl} \\ \rightarrow \mathbf{pat\ is\ a\ girl}$$

(ii) from s1, (i), and r1, infer **pat is a girl**

(iii) from (ii), s2, and r1, infer **pat is human**

Soundness

A formal deduction system **A** is **sound** iff:

$$S \vdash_A F \text{ implies } S \models F$$

Completeness

A formal deduction system **A** is **complete** iff:

$$S \models F \text{ implies } S \vdash_A F$$

Soundness is an essential property – an unsound system is useless (it can be used to prove unjustified relationships).

Completeness is a preferred property – an incomplete system can be useful but cannot prove every logical consequence.

Theories

A theory **T** is a formal system **A** plus a set **S** of formula which are known to be true in some intended interpretation. These formula are called the proper axioms of the theory.

Examples of proper axioms

**S = (pat is a man \vee pat is a woman,
jan is married to pat, etc.**

The intended interpretations are those truth assignments which satisfy all of the proper axioms, e.g.

assignment#1: {pat is a man := True,
pat is a woman := False,
jan is married to pat := True)

assignment#2: {pat is a man := False,
pat is a woman := True,
jan is married to pat := True)

Theorems

A formula **F** is a **theorem** of a theory **T** iff **F** has a formal proof in **T**.

Theoremhood is denoted by:

$$T \vdash F$$

Consistency

A theory **T** is inconsistent iff for some **F**:

$$T \vdash F \quad \text{and} \quad T \vdash \neg F$$

A theory is consistent if it is not inconsistent

Automatic Theorem Proving

Resolution theorem proving

1. Covert all propositions to conjunctive normal form (cnf). For example:

$$((p \vee q) \rightarrow t) \wedge (t \rightarrow s)$$

$$\Rightarrow (\neg(p \vee q) \vee t) \wedge (\neg t \vee s)$$

$$\text{(from } a \rightarrow b = \neg a \vee b \text{)}$$

$$\Rightarrow ((\neg p \wedge \neg q) \vee t) \wedge (\neg t \vee s)$$

$$\text{(from } \neg(a \vee b) = (\neg a \wedge \neg b) \text{)}$$

$$\Rightarrow (\neg p \vee t) \wedge (\neg q \vee t) \wedge (\neg t \vee s)$$

$$\text{(from distributivity of } \vee \text{ over } \wedge \text{)}$$

$$\Rightarrow \{ \{ \neg p, t \}, \{ \neg q, t \}, \{ \neg t, s \} \}$$

$$\text{(using clausal notation)}$$

2. Negate the proposition to be proved convert to cnf, and unite with the cnf form of the original propositions. For example to prove $(p \rightarrow s)$ from the above , negate to give $\neg (p \rightarrow s)$, convert to $\neg (\neg p \vee s)$, then to $(p \wedge \neg s)$, then to $\{\{p\}, \{\neg s\}\}$, then unite with clauses from (1) giving:

$$\{\{\neg p, t\}, \{\neg q, t\}, \{\neg t, s\}, \{p\}, \{\neg s\}\}$$

3. Use the “resolution proof rule” which cancels matching literals $\{a, x, y, \dots\}$ and $\{\neg a, b, c, \dots\} \Rightarrow \{x, y, \dots, b, c, \dots\}$. For example, from the above:

$$\begin{aligned} & \{\{\neg p, t\}, \{\neg q, t\}, \{\neg t, s\}, \{p\}, \{\neg s\}\} \\ \Rightarrow & \{\{\neg q, t\}, \{\neg t, s\}, \{p\}, \{\neg s\}, \{t\}\} \\ \Rightarrow & \{\{\neg q, t\}, \{p\}, \{\neg s\}, \{s\}\} \\ \Rightarrow & \{\{\neg q, t\}, \{p\}, \{\}\} \end{aligned}$$

Proof is complete because the empty clause $\{\}$ has been generated, which signifies an inconsistency, and therefore negating the proposition $(p \rightarrow s)$ to be proven has lead to an inconsistency, so the proposition must be a theorem of the original set of propositions.