Knowledge Representation in the Search for Projective Geometries

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Abstract

Projective geometries have fascinated mathematicians for hundreds of years but a complete characterization of them is still unknown. In this extended abstract we outline some recent searches for finite projective geometries that were accomplished by representing the problem in Boolean logic and then employing satisfiability solvers to complete the search.

Introduction

Projective geometry was first developed by Renaissance artists in order to describe how to project a three dimensional scene onto a two dimensional canvas. Any two lines in a projective geometry must meet—for example, a pair of train tracks (parallel lines in three dimensions) will meet on the horizon when projected onto two dimensions.

A geometry is said to be *finite* if it contains a finite number of points. All finite projective geometries have been completely classified with the exception of those having exactly two dimensions—i.e., the *projective planes* (Cohn 2004). Each line of a projective plane must contain the same number of points and the plane is said to be of *order* n when each line contains $n + 1$ points. The first case for which it is theoretically uncertain if a projective plane of order n exists is for $n = 10$. However, exhaustive computer searches in the 1970s and 1980s eliminated the possibility of this case (Lam 1991).

The resolution of the order ten case required months of computational time using many custom-written programs that were run on the fastest supercomputers of the era. Because the software and hardware are no longer available it is impossible to verify that these searches ran to completion. Moreover, writing custom programs is an inherently error-ridden process (Lam 1990). Indeed, cases missing in the original search were later uncovered (Roy 2011) in an independent check—a check which itself was recently found to miss some partial solutions (Bright et al. 2020a).

Knowledge Representation

By representing the problem in Boolean logic we are able to provide a more verifiable proof. A Boolean logic representation opens the door to using a satisfiability (SAT) solver to perform the search. SAT solvers are very efficient and well-tested pieces of software, so this is generally less error-prone than using custom software. Furthermore, their output does not have to be taken on faith—they produce certificates of unsatisfiability that can be verified using a proof checker.

In order to represent the existence problem in Boolean logic we use the concept of an incidence matrix. The *incidence matrix* of a projective plane is a $\{0, 1\}$ matrix A encoding which points lie on which lines. Because any two lines in a projective plane meet exactly once we know that every off-diagonal entry of AA^T must be exactly 1.

To represent this in Boolean logic, let $a_{i,k}$ be a Boolean variable that is true exactly when the (i, k)th entry of A contains a 1. The fact that the ith and jth lines meet *at most* once can be represented in Boolean logic as

$$\bigwedge_{k,l}(\neg a_{i,k} \lor \neg a_{i,l} \lor \neg a_{j,k} \lor \neg a_{j,l}).$$

and the fact that the ith and jth lines meet *at least* once can be represented in Boolean logic as $\bigvee_k(a_{i,k} \land a_{j,k})$. We use formulae of these forms for all pairs (i, j) with $i < j$.

When $n = 10$ this SAT instance is very difficult to solve but it can be made tractable by exploiting some sophisticated mathematical properties. For example, the error-correcting code associated with A must contain certain codewords that place severe restrictions on the structure of A. For example, the code containing a codeword of Hamming weight 16 implies that A has one of ten possible forms (Carter 1974).

Contributions

We derived a SAT instance for each of the ten possible forms and showed that each case does not have a solution, thereby showing that A's code does not contain codewords of weight 16 (Bright et al. 2020b). The SAT instances were solved in about 30 hours—significantly improving on previous searches which required up to 16,000 hours on modern hardware (Roy 2011; Lam, Thiel, and Swiercz 1986).

Together with the result that A's code does not contain codewords of weight 15 (MacWilliams, Sloane, and Thompson 1973) this implies that A's code must contain codewords of weight 19. This case is more challenging due to some structural differences. However, we are currently solving these SAT instances (Bright, Nejati, and Ganesh 2020) and thereby generating a complete SAT-based verification of the nonexistence of a projective plane of order ten.

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